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EIGENPAIRS OF A FAMILY OF TRIDIAGONAL MATRICES: THREE DECADES LATER

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Abstract. This survey paper summarizes the more important recent applications of the eigenpairs formulas for a family of tridiagonal matrices based on Losonczi's seminal work of almost thirty years ago, which not only seems to have been largely ignored, but has also been re-cast or re-discovered in alternative guises by various authors since. In the course of presenting these applications, we also make contact with earlier more specific applications where Losonczi's work could have been applied to yield the results more quickly. Many of the recent applications in physics and engineering cite less general work, which followed Losonczi more than a decade later.

1. Introduction

For almost a century there has been an active interest/fascination in tridiagonal matrices. For the first 75 years since the pioneering paper of Egerváry and Szász [13], the interest was primarily mathematical where the aim was to develop the spectral theory, viz. the determinants, eigenvalues and eigenvectors, for increasingly more general and complicated tridiagonal matrices. This culminated in the most general class of these matrices studied by Losonczi in [33]. As we shall observe here, over the last two decades there has been an explosion in applications of tridiagonal matrices to important topics in applied mathematics, physics and technology/engineering. In fact, we cannot do due justice to all these applications in this brief survey.

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Thus we have been restricted to presenting only a sample of them. Unfortunately, most of these applications refer to the more recent work of Yueh [52] without recognizing the more general and powerful work of Losonczi that preceded a decade earlier. Even Yueh did not cite Losonczi's work. The aim of the present paper is to highlight the issue in order to give Losonczi the recognition he justly deserves.

2. A class of tridiagonal matrices

Based on an earlier work [34], Losonczi [33] in 1990 introduced and studied the following general family of $n \times n$ complex tridiagonal matrices:

$$(1) \qquad M_{n,k} = \left(\begin{array}{ccc|cccccc}
a & & & c & & & & & \\
& \ddots & & & \ddots & & & & \\
& & a & & & \ddots & & & \\ \cline{1-3}
d & & & 0 & & & \ddots & & \\
& \ddots & & & \ddots & & & \ddots & \\
& & \ddots & & & 0 & & & c \\ \cline{7-9}
& & & \ddots & & & \multicolumn{1}{|c}{b} & & \\
& & & & \ddots & & \multicolumn{1}{|c}{} & \ddots & \\
& & & & & d & \multicolumn{1}{|c}{} & & b
\end{array}\right)_{n\times n} .$$

In this form where $n - 2k \geqslant 0$, the upper left and bottom right corners represent $k \times k$ principal submatrices. For the case of $n - 2k < 0$, the matrices took the following form:

$$(2) \qquad M_{n,k} = \left(\begin{array}{ccc|ccc|ccc}
a & & & & & & c & & \\
& \ddots & & & & & & \ddots & \\
& & a & & & & & & c \\ \hline
& & & a+b & & & & & \\
& & & & \ddots & & & & \\
& & & & & a+b & & & \\ \hline
d & & & & & & b & & \\
& \ddots & & & & & & \ddots & \\
& & d & & & & & & b
\end{array}\right)_{n\times n} ,$$

where the principal submatrices in the upper left and lower right corners are now $(n - k) \times (n - k)$. The main reason for studying these matrices is that they arise: (i) in determining the discrete quadratic inequalities of the Wirtinger type [15] and (ii) when estimating specific coefficients in trigonometric polynomials [13,21,43]. From this study Losonczi was able to derive general formulas for the eigenpairs of $M_{n,k}$, resulting in explicit formulas for several special cases. In what follows we shall without loss of generality only consider matrices of the first form, viz. (1). The corresponding results for matrices of the form in (2) follow naturally and are left as an exercise for the reader.

If we put $n = qk + r$, with $r \in \{0, 1, \ldots, k-1\}$, and let $P_{n,k}(x)$ represent the characteristic polynomial of $M_{n,k}$, then according to Theorem 1 in [33], we find that

$$P_{n,k}(x) = P_{q+1,1}(x)^r \, P_{q,1}(x)^{k-r} \,. \tag{3}$$

This means that the eigenvalues of $M_{n,k}$ can be found from the spectral analysis of the following matrix:

$$M_{\ell,1} = \begin{pmatrix} a & c & & & & \\ d & 0 & c & & & \\ & d & \ddots & \ddots & & \\ & & \ddots & \ddots & c & \\ & & & d & 0 & c \\ & & & & d & b \end{pmatrix}_{\ell\times\ell} . \tag{4}$$

Moreover, from (7) in Theorem 2 of the same reference, we have

$$P_{n,1}(x) = (\sqrt{cd})^n \Big(U_n\Big(\frac{x}{2\sqrt{cd}}\Big) - \frac{a+b}{\sqrt{cd}}\, U_{n-1}\Big(\frac{x}{2\sqrt{cd}}\Big) + \frac{ab}{cd}\, U_{n-2}\Big(\frac{x}{2\sqrt{cd}}\Big)\Big). \tag{5}$$

where $\{U_n\}_{n\geqslant 0}$ are Chebyshev polynomials of the second kind. For $n \ge 1$, these orthogonal polynomials, which have been studied extensively since their discovery in the 1850's (for example, see [10,42]), satisfy the three-term recurrence relation given by

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x)\,, \quad \text{for } n \geqslant 1, \tag{6}$$

subject to the initial values $U_0(x) = 1$ and $U_1(x) = 2x$. One of the most famous formulas for $U_n(x)$, valid for all values of n, is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\ ,$$

where $x = \cos\theta$ and $0 \leqslant \theta < \pi$.

In regard to eigenvectors, if $\lambda = 2\sqrt{cd}\cos\theta$ represents an eigenvalue of $M_{n,1}$, then according to Theorem 3 of [33] the corresponding eigenvectors can be expressed as $u = (u_0, u_1, \ldots, u_{n-1})$, where

$$u_\ell = \Big(\frac{\sqrt{c}}{\sqrt{d}}\Big)^\ell \Big(\sin(\ell+1)\theta + \frac{a}{b}\sin\ell\theta\Big), \quad \text{for } \ell = 0, 1, \ldots, n-1.$$

We present here a survey of some of the most important recent papers that deal with the applications and theory of these tridiagonal matrices, which continue to this day to fascinate scientists in engineering and physics. As part of this survey, we shall discuss important references preceding Losonczi's seminal work as far back as 1928, which deal with more specific or less general matrices than those studied by Losonczi. At the same time we shall cite applications up to the present day in order to demonstrate that these matrices represent a hive of activity in different fields of science and technology. It should be emphasised that the list of publications provided in this survey is by no means exhaustive, but should give the reader a taste of just how important Losonczi's work is.

In the following sections we shall survey several earlier papers containing similar results to Losonczi, then present several other versions of his work, where the authors were unaware of [33] and finally discuss the most recent applications, where the results of [33] emerge in different fields of activity, but are not duly acknowledged. Before doing so, however, we need to make some preliminary remarks to help the reader to understand the later material.

3. Preliminary remarks

First, it is obvious that spectral theory when applied to (1) and especially (4) reduces them to the symmetric case where b and c are replaced by $\sqrt{bc}$. Moreover, if we regard a tridiagonal matrix as the adjacency matrix of a path, then the off-diagonal entries become the weights of each edge. Similarly, one can regard (1) as the (weight) adjacency matrix of two types of path. For more details, the reader is referred to [19].

Second, when we replace d by 1 and c by cd in (1), the resulting matrix is known in the theory of orthogonal polynomials theory as a monic Jacobi matrix [4]. Obviously, the characteristic polynomial is invariant, while the signs of c and d are completely redundant. Nevertheless, they are retained in many examples here for convenience.

4. Before Losonczi

In 1945 Rutherford [44] derived (5) for the case of $c = d = 1$ with a and b, arbitrary. As a consequence, he was able to obtain the eigenvalues for

$$\begin{pmatrix} 1 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}_{n\times n}. \tag{7}$$

These are in turn given by

$$\lambda_\ell = -2\cos\Big(\frac{2\ell\pi}{2n+1}\Big),$$

where $\ell = 1, 2, \ldots, n$. For the matrix

$$\begin{pmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix}_{n\times n} \tag{8}$$

he found that the eigenvalues are given by

$$\lambda_\ell = 2\cos\Big(\frac{\ell\pi}{n}\Big), \quad \text{for } \ell = 1, 2\ldots, n.$$

Several years later, in his Master's thesis [14] Elliott not only re-derived Rutherford's results by providing a more detailed exposition, but also included the matrix

$$\begin{pmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & -1 \end{pmatrix}_{n\times n}, \tag{9}$$

whose eigenvalues were found to be

$$\lambda_\ell = 2\cos\Big(\frac{(2\ell-1)\pi}{2n}\Big), \quad \text{for } \ell = 1, 2\ldots, n.$$

Although outside the scope of this work, it should be pointed out that the matrices studied by Rutherford in [44] were motivated by various problems in physics and chemistry and have since had a tremendous impact in modern research areas such as nanotechnology.

Another motivation for Losonczi's study of his more general tridiagonal matrices was the pioneering paper of E. Egerváry and O. Szász [13] on multivariable operator theory, free pluriharmonic functions and trigonometric polynomials. In this work the authors related their study to the case of $a = b = 0$ in $M_{n,k}$. As a consequence, they derived (3) by rearranging the rows and columns.

Losonczi was able to generalize all the above-mentioned results via one elegant and powerful tridiagonal matrix. Moreover, in addition to calculating the eigenvalues for the matrix, he was able to derive formulas for the corresponding eigenvectors. In the following sections we shall turn our attention to more recent literature, where his results have emerged, but have not been cited. It should be mentioned that the list of applications presented here is by no means exhaustive.

5. Applications

Matrices of the form of (7) are structurally simple but are, nevertheless, rich in applications. Our survey of important examples begins with the description of a major result in [5]. There the authors determine the singular eigenvalues of a Jordan block denoted by $J_n(\mu)$. To determine these values, they present the product of Jordan blocks, one of which is the complex conjugate. Specifically, they consider

$$J_n^*(\mu)J_n(\mu) = \begin{pmatrix} r^2 & r & & \\ r & 1+r^2 & \ddots & \\ & \ddots & \ddots & r \\ & & r & 1+r^2 \end{pmatrix}_{n\times n},$$

where $\mu = re^{i\theta}$. The singular eigenvalues of $J_n(1)$, which represent the main result of their paper, are later found to be

$$\lambda_n = \sqrt{2 + 2\cos\Bigl(\frac{2k\pi}{2n+1}\Bigr)} = 2\cos\Bigl(\frac{k\pi}{2n+1}\Bigr), \quad \text{for } k = 1, \dots, n.$$

This result appears in Theorem 3.3 of [5].

Capparelli and Maroscia also study the generating functions for the determinants of these matrices, but this is outside the scope of the present paper. However, it should be noted that the polynomials associated with matrices of the form given by (4) with $b = 0$, are referred to as co-recursive in the literature, and their properties, particularly orthogonality, have been studied for a long time since Chihara [11]. A recent example is the application of Caputo–Fabrizio fractional operators in various physical problems [9]. The above matrix for $\mu = 1$, viz. $J_n^*(1)J_n(1)$, also appears in many different areas such as civil engineering construction [46,48,49], mechanical engineering [28], or in the analysis of multiconductor transmission lines [25].

Similar matrices also appear in spectral graph theory [1] where we have

$$\begin{pmatrix} 3 & 1 & & \\ 1 & 2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{pmatrix}.$$

Such matrices emerge in deterministic port-based teleportation (dPBT) protocol schemes, where a quantum state is guaranteed to be transferred to another system without unitary correction as in [36]. Moreover, very similar matrices arise in mechanical engineering processes such as freeze-form extrusion fabrication processes [32].

6. Other applications

In the process of linearizing stochastic resonance problems, which arises in a very wide class of dynamical systems, one encounters the following matrix:

$$\begin{pmatrix} \lambda - D & D & & & & \\ D & \lambda - 2D & D & & & \\ & D & \ddots & \ddots & & \\ & & \ddots & \ddots & D & \\ & & & D & \lambda - 2D & D \\ & & & & D & \lambda - D \end{pmatrix}_{n\times n}.$$

The above result is referred to as a tridiagonal *Toeplitz* matrix in [38]. Via spectral theory, the above matrix becomes the type of matrix given by (8). In addition, almost identical matrices can be found in: (i) the theory of experiments in fluids [45] with $D = 1$, (ii) the fast two-dimensional smoothing with the discrete cosine transform in information science [35], where $D = 1$ and $\lambda = 0$, and (iii) the stability of the matrix formulations for the Crank-Nicolson finite difference method for time-dependent diffusion on a staggered grid [39]. In the last example it is found that the above matrix arises when implicit boundary conditions are applied resulting in (4.4) of their paper, which in turn corresponds to $D = 1/2$ and $\lambda = 2$ in the above matrix. To solve the eigenvalue problem these authors use the paper by Yueh [52], which is discussed in the following section, but is itself a particular case of Losonczi's more universal work.

Another application where the above tridiagonal matrices arise is in the study of powers of them such as (8) or (9). This was first carried out by Rimas in [40,41], but continues to fascinate many researchers to this day. Because the knowledge of the eigenpairs of these matrices is vital in these efforts, it means that Losonczi's results in [33] are continually being rediscovered and used. However, he is often not credited for the solutions because the authors are unaware of his seminal work. Instead they only cite more recent references, which also do not cite him properly. In fact, there is a host of papers with applications where the above matrices appear. Some recent examples are [2,7,8,27,30]. Yet again, none of these papers cites Losonczi's work.

7. More general cases

One of the most famous realizations of the general case (4), bearing in mind that it is a simplification of the Losonczi class given by (1), is the Kac–Murdock–Szegő, KMS for short, or Markovian matrix, which first appeared in [29]. This matrix is a square Toeplitz matrix and is defined as

$$K_n(\rho) = \left(\rho^{|i-j|}\right)_{i,j=1,\dots,n},$$

where ρ is a real number. The above class of matrices represents a very fertile area of research and consequently, they continue to be much studied up to the present day. For example, see [12,50] or Problem 7.2 on pp. 12–13 of [26]. The determinants for these matrices are known to be given by $\det K_n(\rho) = (1-\rho^2)^{n-1}$. Furthermore, for $\rho \neq \pm 1$, they reduce to

$$K_n^{-1}(\rho) = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho & & & & \\ -\rho & 1+\rho^2 & -\rho & & & \\ & -\rho & \ddots & \ddots & & \\ & & \ddots & \ddots & -\rho & \\ & & & -\rho & 1+\rho^2 & -\rho \\ & & & & -\rho & 1 \end{pmatrix}_{n\times n}.$$

By discarding the factor outside the above matrix and setting $\rho = 1$, one obtains the matrix $D^T D$, which arises in the theory of signal processing as exemplified by [16].

The general matrix given by (4) is fundamental in the analysis of the KMS matrices. For example, the following tridiagonal matrix appears in [50]:

$$\begin{pmatrix} \frac{2-\gamma}{1-\gamma} & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}_{n\times n},$$

as does the inverse of the matrix

$$L_n = (\min(i-j)-\gamma)_{i,j=1,\ldots,n} .$$

A similar matrix also appears in [17].

At the beginning of this section we presented the type of matrix that emerges in the linearization of stochastic resonance problems given by (6). In their study of the stochastic resonance problem modelled as a one-dimensional array of nonlinear spatially coupled subsystems in the mean field limit, Nicolis and Nicolis [37] obtain the following matrix:

$$\begin{pmatrix} \lambda-D & D & & & & \\ D & -2\lambda-D & D & & & \\ & D & \ddots & \ddots & & \\ & & \ddots & \ddots & D & \\ & & & D & -2\lambda-D & D \\ & & & & D & \lambda-D \end{pmatrix}_{n\times n}.$$

In determining the eigenvalues for this matrix they cite once again the work of Yueh [52], which as have stated previously did not reference the earlier and more general work by Losonczi.

Occasionally, due to awkward notation, it is not obvious that simple tridigonal matrices of the above form emerge. For example, in the diagonalization of the Ising model with a transverse magnetic field in the presence of a local field defect at one edge, Francica et al. obtain by introducing the nonlocal Jordan-Wigner tranformation the following matrix in Appendix A of [20]:

$$\begin{pmatrix} 1+h^2\mu^2 & -h & & & \\ -h & 1+h^2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+h^2 & -h \\ & & & -h & h^2 \end{pmatrix}.$$

Next the authors obtain a version of (5) for their specific application. Again, rather than cite Losonczi, they use the Yueh's approach to derive (A12) in their paper. Interestingly, if one sets $\mu = 0$ in the above matrix, then one obtains (B3) in [22], which deals with the study of freeze-in dark matter models where the "Clockwork" mechanism is used to suppress dark matter couplings. Again, Yueh's paper is cited in their derivation of the eigenvalues for the fermionic Clockwork matrix.

Another physical example of this type of matrix is (6) in [3], which takes the following form:

$$\begin{pmatrix} -2\Delta_L & 2J_L & & & \\ 2J_L & -4\Delta_L & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & -4\Delta_L & 2J_L \\ & & & 2J_L & -2\Delta_L \end{pmatrix}.$$

Balachandran et al. refer to this as a Toeplitz bordered matrix. It arises from studying two spin chains of the Heisenberg XXZ model of condensed matter physics, each with a different anisotropy parameter. To explain the spin current rectification arising from the different anisotropy in both half-chains, they consider the case of coupling two chains, one of which is completely polarized and the other is at infinite temperature. By studying the case where the half-chain with non-zero anisotropy is polarized, they derive the above matrix determining its eigenvalues, which represents the magnon excitation spectrum for the half-chain.

In a similar fashion Guo and Poletti [23,24] also obtain an $n \times n$ tridiagonal Toeplitz bordered matrix in their solution for a large class of Lindblad master equations for noninteracting particles (both fermions and bosons are considered) on n sites. For the fermionic case the tridiagonal matrix is given by

$$\begin{pmatrix} -ih_z - \Gamma_1/2 & -iJ/2 & & & \\ -iJ/2 & -ih_z & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & -ih_z & -iJ/2 \\ & & & -iJ/2 & -ih_z - \Gamma_n/2 \end{pmatrix}.$$

Once again, the eigenvalues are determined by referring to the more recent work of Yueh rather than Losonczi's work.

8. Extensions

As described in [19], the so-called k-tridiagonal matrices denoted as $T_n^{(k)}(a_*, b_*, c_*)$ with $a_* \doteqdot \{a_1, \dots, a_n\}$, $b_* \doteqdot \{b_1, \dots, b_{n-k}\}$ and $c_* \doteqdot \{c_{k+1}, \dots, c_n\}$ can be interpreted as a matrix, whose graph is a set of disjoint paths or a forest with r paths of length $\ell - 1$ and $k - r$ paths of length ℓ, where $n = k\ell - r$. The authors then show that these matrices can be expressed as a direct sum of tridiagonal matrices denoted by T_n. Consequently, the application of spectral theory, i.e., solving for the determinant, eigenvalues and eigenvectors, of these more general matrices than those discussed in Sections 4 to 6 can be achieved by using Egerváry and Szász [13] or the more general results of Losonczi [33].

There are two interesting extensions of these matrices. The first is when the main diagonal is comprised of zero entries and both subdiagonals are not in symmetric positions. Such matrices are studied in [31]. The other has no such restriction on the main diagonal, but possesses a quasi-symmetry in the sense that the upper subdiagonal begins horizontally at position $k + 1$, and the lower subdiagonal begins vertically at $k + 2$ as described in [47]. There they are referred to $(k, k + 1)$ tridiagonal matrices of order n and are denoted by $T_n^{(k,k+1)}$ with $n \le 2k$.

The second type of generalization includes matrices of the form

$$\begin{pmatrix} a & c & & & & \\ c & 0 & e & & & \\ & e & \ddots & \ddots & & \\ & & \ddots & \ddots & e & \\ & & & e & 0 & c \\ & & & & c & b \end{pmatrix}.$$

The above matrices are studied in detail in [6,18] with many specific cases of this extension together with their spectra. The eigenvalues are derived by relying on the frequently cited [51,52] which we have indicated are simply replicating Losonczi's results.

9. Conclusion

In this brief survey, we have presented a great number of important applications/problems in mathematics, physics and technology involving tridiagonal matrices that fall into the class given by (1). The spectral theory for these matrices and those given by (2) was developed by Losonczi in the early 1990's [33]. Yet this work has been largely ignored. In fact, many of the more recent problems cited here have been solved with reference to Yueh's work [52], which appeared more than a decade later and is less general than Losonczi's paper. It is hoped that this article has redressed this issue so that Losonczi's work receives the due recognition it deserves.

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