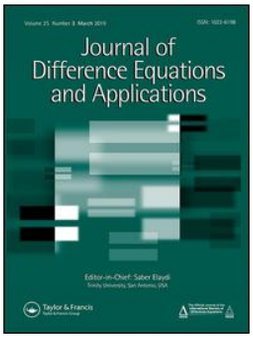

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A matrix approach to some second-order difference equations with sign-alternating coefficients

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ABSTRACT
In this paper, we analyse and unify some recent results on the double sequence $\{y_{n,k}\}$, for $n, k \geqslant 1$, defined by the second-order difference equation

$$y_{n,k} = (-1)^{\lfloor (n-1)/k \rfloor} y_{n-1,k} - y_{n-2,k},$$

with $y_{1,k} = 1$ and $y_{2,k} = 0$, in terms of matrix theory and orthogonal polynomials theory. Moreover, we provide a general solution to

$$z_{n,k} = (-1)^{\lfloor (n-1)/k \rfloor} z_{n-1,k} - (-1)^{\lfloor (n-2)/a \rfloor} z_{n-2,k},$$

using a closely related approach. We discuss briefly other recent problems involving a general recurrence relation of second order and relate them with the existing literature.

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1. Preliminaries

The scrutiny of the general recurrence relation of second order

$$u_{n+1} = a\, u_n + b\, u_{n-1}, \tag{1}$$

with certain initial conditions, goes back to 1960s with the study of the algebraic properties and relations of the sequence (u_n) [2, 15]. Most of the notorious number sequences are obtained from (1), namely the Fibonacci numbers, with $a = b = u_0 + 1 = u_1 = 1$, the Lucas numbers, when $a = b = u_0 - 1 = u_1 = 1$, or the Pell numbers, for $a - 1 = b = u_0 + 1 = u_1 = 1$.

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Each u_n defined by (1) can be obtained from the determinant of the monic Jacobi matrix

$$T_n = \begin{pmatrix} a & 1 & & \\ -b & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -b & a \end{pmatrix}_{n\times n} \tag{2}$$

or, as it has become popular, the permanent of

$$\begin{pmatrix} a & 1 & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & b & a \end{pmatrix}_{n\times n},$$

and perturbations on some diagonal/subdiagonal entries together with the specialization of the initial conditions. For instance, the Fibonacci numbers can be obtained directly from

$$\det \begin{pmatrix} 1 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 1 \end{pmatrix}_{n\times n} \quad \text{or, equivalently,} \quad \det \begin{pmatrix} 1 & \mathbb{i} & & \\ \mathbb{i} & \ddots & \ddots & \\ & \ddots & \ddots & \mathbb{i} \\ & & \mathbb{i} & 1 \end{pmatrix}_{n\times n},$$

where $\mathbb{i}$ represents the unit imaginary number, with $a = b = 1$ in (2) and $u_0 + 1 = u_1 = 1$ (cf. [5, 20]).

At the same time, from the well-established theory of orthogonal polynomials (see, e.g. the classical reference [3]), the determinant (2) can be given by (cf. [10])

$$\det T_n = (-\mathbb{i}\sqrt{b})^n \, U_n\left(\frac{a\mathbb{i}}{2\sqrt{b}}\right),$$

where $\{U_n(x)\}_{n\geqslant 0}$ are the Chebyshev polynomials of the second kind, i.e. the orthogonal polynomials satisfying the three-term recurrence relations

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0, \quad \text{for } n = 1, 2, \ldots,$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$ (cf. [4, p.81]). Among all formulas for the Chebyshev polynomials of the second kind, perhaps the best known is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0, 1, 2 \ldots$ (cf. [4, p.413]). Notwithstanding, any explicit formula for $U_n(x)$ is not relevant for our purposes.

This means that, the nth Fibonacci number F_n can be given by (cf. [2, 15])

$$F_n = (-\mathbb{i})^{n-1}\, U_{n-1}\left(\frac{\mathbb{i}}{2}\right), \tag{3}$$

while the nth Lucas number L_n is

$$L_n = (-\mathbb{i})^{n}\left(U_n\left(\frac{\mathbb{i}}{2}\right) - U_{n-2}\left(\frac{\mathbb{i}}{2}\right)\right).$$

Regarding the Pell numbers P_n's, we have elegant expressions as

$$P_n = (-\mathbb{i})^{n-1}\, U_{n-1}(\mathbb{i}),$$

for all $n = 1, 2, 3, \ldots$, since $(-\mathbb{i})^{-1} U_{-1}(\mathbb{i}) = 0$, $(-\mathbb{i})^{0}\, U_0(\mathbb{i}) = 1$, and

$$(-\mathbb{i})^{n+1}\, U_{n+1}(\mathbb{i}) = 2(-\mathbb{i})^{n}\, U_n(\mathbb{i}) + (-\mathbb{i})^{n-1}\, U_{n-1}(\mathbb{i}).$$

For more sophisticated sequences, as the Lucas–Pell numbers, LP_n, which is a derivation of the Pell sequence setting the initial conditions $u_0 = u_1 = 2$, we have

$$LP_n = (-\mathbb{i})^{n-1}\ \left(U_{n-1}(\mathbb{i}) - U_{n-3}(\mathbb{i})\right),$$

or the Jacobsthal numbers J_n's where $a = 1$ and $b = 2$ and initial conditions $u_0 = 0$ and $u_1 = 1$, where we simply have

$$J_n = (-\mathbb{i}\sqrt{2})^{n-1} U_{n-1}\left(\frac{\mathbb{i}}{2\sqrt{2}}\right).$$

We remark that when we want to work on determinants of Jacobi matrices, all considerations can be reduced to the analysis of the monic case, since the determinant (or the permanent) depends on the product of each subdiagonal entry with its symmetric counterpart. So we can avoid trivial cases as the one considered in [25]. The case when $a = -1$ and $b = 1$ has been considered in [16]. In this concrete case, we simply get $\det T_n = \mathbb{i}^{n-1} U_{n-1}(\mathbb{i}/2) = (-1)^{n-1} F_n$.

A much harder problem results when the coefficients in (1) are non-constant. That was recently the subject of particular interest when they are sign alternated in [18, 23–25]. Our aim here is to put together these and several other recent results achieved in different settings to one place and in a common framework. We will also provide an answer to the questions left open in [23]. Our approach is constructive in the sense that no induction will be used. First, we make a few considerations on recent results regarding a particular case of (1), recalling some less known facts.

2. Fibonacci *k*-numbers

The case when $b = 1$ in (1) has been thoroughly studied for the past decade with a vast multitude of results appearing in the literature. Many of them, can be deduce from the existing classic theory. These numbers were coined as k-Fibonacci numbers [7] or, quite

recently, as l-numbers [9], making $a = k$ or l, respectively. In this section, we briefly discuss these two references.

First, we note that for T_n defined in (2), we have the standard identity

$$\det T_n = \det T_k(x) \det T_\ell(x) + b \det T_{k-1}(x) \det T_{\ell-1}(x), \tag{4}$$

when $k + \ell = n$. Indeed this identity can be established for any tridiagonal matrix and that will be useful later on. It can also be seen as a generalization of the standard recursive formula

$$\det T_n = a \det T_{n-1}(x) + b \det T_{n-2}(x).$$

From (4), we can easily deduce [9, Proposition 2.2 (2)] or [7, Proposition 14] as well as $u_n^2 + u_{n+1}^2 = u_{2n+1}$, setting $k = \ell = n$ (cf. [9, Remark 2.1 (1)]). Moreover, we can also write

$$u_{n+4} = (l^2 + b)u_{n+2} + lu_{n+1},$$

which, from the definition (1), means

$$u_{n+4} = (l^2 + b + 1)u_{n+2} - bu_n.$$

This is the general case for [9, Proposition 2.3 (1)], and the knowledge of any explicit expression is not required.

As for [9, Proposition 2.2 (1)], which states that if d divides n, then u_d divides u_n, it can be derived from [19, Theorem 3]. Actually, this theorem is even stronger since it provides and equivalence between the two conditions. Moreover, this theorem also allows to provide precisely the remainder of the Euclidean division of two l-numbers, which is another l-number.

On the other hand and perhaps more interesting, we can also conclude from [19, Theorem 4] that, if u_n and u_m are two so-called l-numbers and $g = \gcd(m, n)$, then

$$\gcd(u_n, u_m) = u_g,$$

which is a result that, for Fibonacci numbers (i.e. $l = 1$), apparently goes back to Édouard Lucas in 1876.

The Fibonacci k-numbers are other side of the same notion. In this case, the motivation was a recursive application of two geometrical transformations, each considered as a 2×2 matrix, used in the four-triangle longest-edge partition [7]. Some identities explored and rediscovered in [7] deserve a brief account.

Setting $a = k$ and $b = 1$ in (1), we have in general

$$u_{n+1}u_{n-1} - u_n^2 = (ku_n + u_{n-1}) \quad u_{n-1} - (ku_{n-1} + u_{n-2}) \quad u_n = -(u_n u_{n-2} - u_{n-1}^2).$$

Using an inductive reasoning, we conclude [7, Proposition 7]. Similarly, from

$$\frac{1}{k}\left(u_{n+2}+u_{n+1}-1\right)=\frac{1}{k}(u_{n+1}+u_n-1)+u_{n+1},$$

we may easily deduce [7, Proposition 8] along the remaining results. Of course, we can use alternatively the identities

$$\begin{aligned} ku_1 &= u_2-u_0\\ ku_2 &= u_3-u_1\\ ku_3 &= u_4-u_2\\ ku_4 &= u_5-u_3\\ &\vdots \end{aligned}$$

to prove again

$$\sum_{i=1}^{n} u_i=\frac{1}{k}(u_{n+1}+u_n-1).$$

Among the plethora of results on Fibonacci k-numbers, we point out the difference between two consecutive numbers. This was, for example, the topic treated recently in [6], where the sequence $v_n=u_{n+1}-u_n$ was considered. We recall that $a=k$ and $b=1$ in (1). For instance, from [1, Theorem 1], v_n satisfies (1) as well. This is [6, Lemma 1]. On the other hand, since

$$v_n=(k-1)u_n+u_{n-1},$$

we have

$$\begin{aligned} v_{n+1}-v_n &= (k-1)u_n n+1+u_n-(k-1)u_n n-u_{n-1}\\ &= (k-1)v_n+v_{n-1}. \end{aligned}$$

This is exactly what is stated in [6, Theorem 1]. The other main results of [6] can be obtained from a similar strategy as described above.

3. Tridiagonal matrices with biperiodic alternating signs

We turn now our attention to the main aim of this paper. Recently, the determinants of the tridiagonal matrices

$$B_n=\begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & -1 & 1 & \\ & & 1 & -1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}_{n\times n} \tag{5}$$

and

$$C_n = \begin{pmatrix} 1 & 1 & & & & \\ -1 & -1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & & -1 & -1 & 1 & \\ & & & 1 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}_{n\times n} \tag{6}$$

were analysed in [18, 24] in terms of Fibonacci numbers. Multiplying the two diagonal matrices diag $(1, 1, -1, -1, 1, 1, \ldots)$ at left and diag $(1, -1, 1, -1, 1, -1, \ldots)$ at right of B_n, it is not difficult to see that $\det B_n = (-1)^\kappa \det C_n$, where $\kappa = n$, if $n \equiv 0 \pmod 4$, and $\kappa = n - 1$, otherwise. In this section, we find a common induction-free framework to both cases using some results on orthogonal polynomials.

The concept of *tridiagonal k-Toeplitz matrix* – an $n \times n$ complex tridiagonal matrix $A = (a_{ij})$ such that

$$a_{i+k,j+k} = a_{ij}, \quad \text{for } i, j = 1, 2, \ldots, n-k,$$

for a given positive integer number $k < n$ – was introduced independently in [5, 13, 14, 17, 20]. For our purposes, when $k = 2$, we have a matrix

$$A_n = \begin{pmatrix} a_1 & 1 & & & & \\ b_1 & a_2 & 1 & & & \\ & b_2 & a_1 & 1 & & \\ & & b_1 & a_2 & 1 & \\ & & & b_2 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}_{n\times n}. \tag{7}$$

The determinant of A_n defined in (7) can be found for example in [12, Theorem 4.1] in the following way

$$\det A_{2\ell} = (\sqrt{b_1 b_2})^\ell \left(U_\ell \left(\frac{a_1 a_2 - b_1 - b_2}{2\sqrt{b_1 b_2}} \right) + \frac{\sqrt{b_2}}{\sqrt{b_1}} U_{\ell-1} \left(\frac{a_1 a_2 - b_1 - b_2}{2\sqrt{b_1 b_2}} \right) \right) \tag{8}$$

and, otherwise,

$$\det A_{2\ell+1} = a_1 (\sqrt{b_1 b_2})^\ell \, U_\ell \left(\frac{a_1 a_2 - b_1 - b_2}{2\sqrt{b_1 b_2}} \right). \tag{9}$$

So, from (8),

$$\begin{aligned} \det C_{2\ell} &= \mathbb{i}^\ell \left(U_\ell \left(\frac{\mathbb{i}}{2} \right) - \mathbb{i}\, U_{\ell-1} \left(\frac{\mathbb{i}}{2} \right) \right) \\ &= (-1)^k F_{\ell-1} \end{aligned}$$

and, from (9),

$$\begin{aligned}\det C_{2\ell+1} &= \mathbb{i}^{\ell}\, U_{\ell}\left(\frac{\mathbb{i}}{2}\right)\\ &= (-1)^{\ell}\, F_{\ell+1},\end{aligned}$$

taking into account (3). Now both [18, Theorem 2] and [24, Theorem 1] follow.

Remark 3.1: While for $b_1 b_2 > 0$, we can write $\sqrt{b_2/b_1}$ in (8), as in [12, Theorem 4.1], in general one must consider $\sqrt{b_2}/\sqrt{b_1}$. This includes the case when $b_1 b_2 < 0$.

Remark 3.2: A particular instance for $k = 2$ was first studied by D.E. Rutherford [21] in 1945.

4. A generalization

In [23], P. Trojovský provided a solution to the second-order difference equations where the signs of the coefficients change with a defined periodicity (cf. [23, Theorems 1 and 2]). Namely,

$$y_{n,k} = (-1)^{\lfloor (n-1)/k \rfloor} y_{n-1,k} - y_{n-2,k}, \tag{10}$$

for an integer $k > 1$, with initial conditions $y_{1,k} = 1$ and $y_{2,k} = 0$. Here $\lfloor x \rfloor$ represents the integer part of x. Additionally, we use $y_{n,k}$ instead $y_n^{(k)}$ to avoid potential the notation of derivative.

It is clear that to (10) we can associate the determinant of the matrix

$$B_{n,k} = \left(\begin{array}{cccc|cccc|ccc}
1 & 1 & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & \\
 & \ddots & \ddots & 1 & & & & & & & \\
 & & 1 & 1 & 1 & & & & & & \\
\hline
 & & & 1 & -1 & 1 & & & & & \\
 & & & & 1 & \ddots & \ddots & & & & \\
 & & & & & \ddots & \ddots & 1 & & & \\
 & & & & & & 1 & -1 & 1 & & \\
\hline
 & & & & & & & 1 & 1 & 1 & \\
 & & & & & & & & 1 & \ddots & \ddots \\
 & & & & & & & & & \ddots &
\end{array}\right)_{n\times n}, \tag{11}$$

where the first $\lfloor n/k \rfloor$ blocks are of order k. Clearly $B_{n,2} = B_n$, as defined in (5). However, for this case, we need a different strategy in order to provide a compact formula to (10).

We adopt the explicit formula for the determinant of the tridiagonal k-Toeplitz matrix

$$A_n = \begin{pmatrix} a_1 & 1 & & & & & & & \\ b_1 & \ddots & \ddots & & & & & & \\ & \ddots & a_k & 1 & & & & & \\ & & b_k & a_1 & 1 & & & & \\ & & & b_1 & \ddots & \ddots & & & \\ & & & & \ddots & a_k & 1 & & \\ & & & & & b_k & a_1 & 1 & \\ & & & & & & b_1 & \ddots & \ddots \\ & & & & & & & \ddots & \end{pmatrix}_{n\times n}, \tag{12}$$

when $k > 2$, established in [11, Theorem 5.1] (see also [20]). We next briefly state it. For $j \geqslant i$, define

$$\Delta_{i,j} = \det \begin{pmatrix} -a_i & 1 & & \\ b_i & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & b_{j-1} & -a_j \end{pmatrix},$$

and, for $j < i$, set

$$\Delta_{i,j} = \begin{cases} 0 & \text{if } j < i-1 \\ 1 & \text{if } j = i-1 \end{cases}.$$

Let us set now

$$\varphi_k = \frac{1}{2\mu}\left(d_k + (-1)^k (b_k + \mu^2/b_k)\right),$$

where d_k is

$$d_k = \det \begin{pmatrix} -a_1 & 1 & & & 1 \\ b_1 & -a_2 & 1 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ b_k & & & b_{k-1} & -a_k \end{pmatrix}$$

and $\mu = \sqrt{b_1 \cdots b_k}$. Then, for $0 \leqslant r \leqslant k-1$ and $n \equiv r \pmod{k}$, the determinant of A_n is

$$\det A_n = (-1)^n \mu^{\lfloor n/k \rfloor} \left(\Delta_{1,r}\, U_{\lfloor n/k \rfloor}(\varphi_k) + \frac{b_1 \cdots b_r b_k}{\mu}\, \Delta_{r+2,k-1}\, U_{\lfloor (n-k)/k \rfloor}(\varphi_k) \right). \tag{13}$$

If we consider the n-diagonal matrix

$$D_{n,k} = \operatorname{diag}(\underbrace{1,\ldots,1}_{k\times}, \underbrace{-1,\ldots,-1}_{k\,\times}, 1, \ldots),$$

we can easily find that

$$D_{n,k}\,B_{n,k} = \left(\begin{array}{cccc|cccc|cccc}
1 & 1 & & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & & \\
 & \ddots & \ddots & 1 & & & & & & & & \\
 & & 1 & 1 & 1 & & & & & & & \\
\hline
 & & & -1 & 1 & 1 & & & & & & \\
 & & & & 1 & \ddots & \ddots & & & & & \\
 & & & & & \ddots & \ddots & 1 & & & & \\
 & & & & & & 1 & 1 & 1 & & & \\
\hline
 & & & & & & & -1 & 1 & 1 & & \\
 & & & & & & & & 1 & \ddots & \ddots & \\
 & & & & & & & & & \ddots & &
\end{array}\right)_{n\times n}. \tag{14}$$

Let us designate by $\tilde{B}_{n,k}$ the matrix defined in (14). Hence, finding $\det B_{n,k}$ is equivalent to obtaining $\det D_{n,k}$ and $\det \tilde{B}_{n,k}$.

First we deal with $\det D_{n,k}$. Let $\varpi_p^{(k)}$ denote the pth partial sum of the (infinite) sequence

$$s^{(k)} = (\underbrace{0,\ldots,0}_{k\times},\underbrace{1,\ldots,1}_{k\times},0,\ldots). \tag{15}$$

The sequence (15) and $\varpi_p^{(k)}$ are important in many instances. The simplest case when $k=2$ can be found in [22, A021913 and A133872] as a particular case of the lexicographically earliest de Bruijn sequence. In general, we have

$$s^{(k)}(n) = 1 - \lfloor ((n+k-1) \pmod{2k})/k \rfloor, \tag{16}$$

as we can see in [22, A088911].

Regarding the partial sums $\varpi_p^{(k)}$'s, as in [22, A194272], for $k=3$, we can display each sum in an array where tth row is obtained by adding $k\,(t-1)$ to each entry of

$$\underbrace{0\cdots 0}_{k\times}12\cdots k.$$

The parity of each $\varpi_p^{(k)}$ can be easily found as well as

$$\det D_{n,k} = (-1)^{\varpi_p^{(k)}}. \tag{17}$$

We focus now our attention to $\tilde{B}_{n,k}$. Setting $a_1=\cdots=a_k=b_1=\cdots=b_{k-1}=1$ and $b_k=-1$ in (12), then

$$\Delta_{i,j} = U_{j-i+1}\left(-\frac{1}{2}\right) = \begin{cases} -1 & \text{if } j-i\equiv 0 \pmod 3 \\ 0 & \text{if } j-i\equiv 1 \pmod 3 \\ 1 & \text{if } j-i\equiv 2 \pmod 3 \end{cases},$$

and

$$\varphi_k = \frac{d_k}{2\,\mathbb{i}},$$

where d_k is

$$d_k = \begin{cases} 0 & \text{if } k \equiv 0 \pmod 3 \\ -1 & \text{if } k \equiv 1 \pmod 3 \\ 1 & \text{if } k \equiv 2 \pmod 3 \end{cases},$$

since $d_k = \Delta_{2,k} - \Delta_{3,k} - \Delta_{2,k-1} - 2$, from [8, Corollary 2.4]. Consequently,

$$\varphi_k = \begin{cases} 0 & \text{if } k \equiv 0 \pmod 3 \\ \mathbb{i}/2 & \text{if } k \equiv 1 \pmod 3 \\ -\mathbb{i}/2 & \text{if } k \equiv 2 \pmod 3 \end{cases}.$$

In conclusion, for $0 \leqslant r \leqslant k-1$ and $n \equiv r \pmod k$, we have from (13)

$$\det \tilde{B}_{n,k} = (-1)^n\, \mathbb{i}^{\lfloor n/k \rfloor} \left(U_r\left(-\frac{1}{2}\right) U_{\lfloor n/k \rfloor}(\varphi_k) + \mathbb{i}\, U_{k-r-2}\left(-\frac{1}{2}\right) U_{\lfloor (n-k)/k \rfloor}(\varphi_k) \right).$$

Taking into account that

$$U_n\left(-\frac{1}{2}\right) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ -1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3 \end{cases} \quad \text{and} \quad U_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} & \text{if } n \text{ is even} \end{cases},$$

we may conclude the following theorem.

Theorem 4.1: *Let n, k, q, r be integers such that $0 \leqslant r \leqslant k-1$ and $n = qk + r$. Then the determinant of $\tilde{B}_{n,k}$, defined in* (14), *up to the product with* $(-1)^n$, *is* Table 1

Multiplying each equality of Theorem 4.1 by $\det D_{n,k}$ found in (17), we reach [23, Theorem 2].

5. A new periodic difference equation

In [23], P. Trojovský claimed that the solution for the difference equation

$$z_{n,k} = (-1)^{\lfloor (n-1)/k \rfloor} z_{n-1,k} - (-1)^{\lfloor (n-2)/k \rfloor} z_{n-2,k}, \tag{18}$$

Table 1. $\det \tilde{B}_{n,k}$.

	$k \equiv 0 \pmod 3$	$k \equiv 1 \pmod 3$	$k \equiv 2 \pmod 3$
$r \equiv 0 \pmod 3$	1	$(-1)^q F_{q+1}$	F_{q-1}
$r \equiv 1 \pmod 3$	-1	$(-1)^{q+1} F_{q+2}$	$-F_{q+1}$
$r \equiv 2 \pmod 3$	0	$(-1)^q F_q$	F_q

with initial conditions $z_{1,k} = 1$ and $z_{2,k} = 0$, is for $k = 3$

$$z_{n,3} = \begin{cases} F_{\frac{n}{2}+1} & \text{if } n \equiv 0 \pmod 6 \\ F_{n-3\lfloor \frac{n}{6} \rfloor - 1} & \text{if } n \equiv 1, 4 \pmod 6 \\ -F_{n-3\lfloor \frac{n}{6} \rfloor - 1} & \text{if } n \equiv 2, 3, 5 \pmod 6 \end{cases},$$

while, for $k = 2$, we have $z_{n,2} \in \{-2, -1, 0, 1\}$.

Our aim in this section is to provide a full comprehension of this problem, providing a general formula for (18) and analyse these two particular cases.

Our approach now is slightly different from Section 4. The matrix associated to (18) is

$$E_{n,2k} = \left(\begin{array}{cccccc|ccc} 1 & 1 & & & & & & & \\ 1 & \ddots & \ddots & & & & & & \\ & \ddots & 1 & 1 & & & & & \\ & & 1 & -1 & 1 & & & & \\ & & & -1 & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & 1 & & \\ & & & & & -1 & -1 & 1 & \\ \hline & & & & & & -1 & 1 & 1 \\ & & & & & & & 1 & \ddots & \ddots \\ & & & & & & & & \ddots & \end{array}\right)_{n\times n}, \tag{19}$$

where the first $\lfloor n/(2k) \rfloor$ blocks are of order $2k$. The main diagonal and the non-constant subdiagonal repeat periodically

$$(\underbrace{1, \ldots, 1}_{k\times}, \underbrace{-1, \ldots, -1}_{k\times}),$$

and we have an all 1's superdiagonal. This means that

$$\Delta_{i,j} = \begin{cases} U_{j-i+1}\left(\frac{\mathbb{i}}{2}\right) & \text{if } k < i \leqslant j \\ U_{k-i+1}\left(\frac{1}{2}\right) U_{j-k}\left(\frac{\mathbb{i}}{2}\right) - U_{k-i}\left(\frac{1}{2}\right) U_{j-k-1}\left(\frac{\mathbb{i}}{2}\right) & \text{if } i < k \leqslant j \\ U_{j-i+1}\left(\frac{1}{2}\right) & \text{if } i \leqslant j \leqslant k \\ 0 & \text{if } j < i-1 \\ 1 & \text{if } j = i-1 \end{cases}.$$

On the other hand, we have

$$\varphi_{2k} = \frac{(-\mathbb{i})^{\delta_{k,\text{odd}}}}{2}\left(\Delta_{2,2k-1} - \Delta_{2,2k} - \Delta_{3,2k} + 2\delta_{k,\text{even}}\right),$$

where $\delta_{i,j} = 1$, if i and j have the same parity, and 0 otherwise.

For example, when $k = 2$, we have Table 2

Table 2. $z_{n,2}$.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$q \equiv 0 \pmod 3$	1	-1	0	1
$q \equiv 1 \pmod 3$	-2	1	1	0
$q \equiv 2 \pmod 3$	1	0	-1	-1

where $n = 4q + r$, with $0 \leqslant r < 4$. Notice that $\varphi_4 = -1/2$.

Let us take a look now into the case when $k = 3$. Here $\varphi_6 = -2\mathrm{i}$. Assume that $n = 6q + r$, with $0 \leqslant r < 6$ (Table 3).

Table 3. $z_{n,3}$.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
$\Delta_{1,r}$	1	-1	0	1	1	2
$\Delta_{r+2,5}$	1	-3	2	1	1	0
$b_1 \cdots b_r b_6/\mu$	i	i	i	i	$-\mathrm{i}$	i
$z_n^{(3)}$	F_{3q+1}	F_{3q-1}	$-F_{3q}$	$-F_{3q+1}$	F_{3q+2}	$-F_{3q+3}$

These two tables show that Trojovský's final assertions in [23] were indeed correct.

We may wonder now what happens for any order n. We will provide now the solution for $k = 4$ and $k = 5$ (Table 4).

Table 4. Factors for $z_{n,4}$.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$\Delta_{1,r}$	1	-1	0	1	-1	-2	-3	-5
$\Delta_{r+2,7}$	3	2	-5	3	2	1	1	0
$b_1 \cdots b_r$	1	1	1	1	1	-1	1	-1

Now $z_n^{(4)}$ is given by

$$(-1)^n \left(\Delta_{1,r}\, U_q\left(-\frac{5}{2}\right) - b_1 \cdots b_r\, \Delta_{r+2,7}\, U_{q-1}\left(-\frac{5}{2}\right) \right),$$

where q and r are integers numbers defined as above. It is interesting to observe the first terms of the sequence $(U_n(\frac{5}{2}))$ are

$$5,\ 24,\ 115,\ 551,\ 2640,\ 12,649,\ 60,605,\ 290,376,\ 1,391,275,\ 6,665,999,\ 31,938,720,$$

which indicates that this is the sequence [22, A004254].

As for $z_{n,5}$ we have Table 5

Table 5. Factors for $z_{n,5}$.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$
$\Delta_{1,r}$	1	-1	0	1	-1	0	1	1	2	3
$\Delta_{r+2,9}$	-8	5	3	-8	5	3	2	1	1	0
$b_1 \cdots b_r$	1	1	1	1	1	1	-1	1	-1	1

Then, $z_{n,5}$ is given by

$$(-1)^n \, \mathbb{i}^q \left(\Delta_{1,r} \, U_q \left(\frac{3\mathbb{i}}{2} \right) + b_1 \cdots b_r \mathbb{i} \, \Delta_{r+2,9} \, U_{q-1} \left(\frac{3\mathbb{i}}{2} \right) \right).$$

Here, the sequence $((-\mathbb{i})^n U_n(3\mathbb{i}/2))$ is the so-called 'bronze Fibonacci numbers' (cf. [22, A006190]) and its first terms are

$$3, \ 10, \ 33, \ 109, \ 360, \ 1189, \ 3927, \ 12,970, \ 42,837, \ 141,481.$$

6. An open problem

The remaining case of these difference equations with alternating signs to study is when

$$x_{n,k} = x_{n-1,k} - (-1)^{\lfloor (n-2)/k \rfloor} x_{n-2,k},$$

with initial conditions $x_{1,k} = 1$ and $x_{2,k} = 0$. We believe that a solution can be given following our approach with no need of induction. We leave it as an open problem.

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